HANDLE REMOVAL ON A COMPACT RIEMANN SURFACE OF GENUS 2

BY

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ABSTRACT

A degeneration of a compact, two-sheeted, Riemann surface in genus 2 is studied. Two branch points coalesce when the period matrix for the surface is degenerated. A Siegel modular transformation is applied to give the degenerating period matrix in the form of a corner matrix.

I.

The purpose of this note is to obtain a result complementing the work in [1]. In [1], a compact Riemann surface of genus 2 was split into two surfaces, each of genus 1, by means of a specified prescription. Here we change the prescription in order to make the Riemann surface degenerate by dropping a handle. The reader is referred to [1] for notation and introductory material.

II.

We define a degeneration of the Riemann surface S with period matrix (π_{ij}) by letting π_{22} tend to $i\infty$. We start with a surface which has the branch points $1/\lambda_1$ and $1/\lambda_2$ near each other, far from all other branch points and with $|\pi_{22}|$ large. We make the assumption that with the given homology basis, $|\pi_{22}|$ will indeed be large when $1/\lambda_1$ and $1/\lambda_2$ are near one another.

THEOREM 1. *Let S degenerate as described in the above definition. Then S degenerates to a Riemann surface in genus* 1 *with branch points over* $0, 1, \infty$ *and* $1/\lambda_3$.

PROOF. We examine the theta constants appearing in the formulae for the three moveable branch points given in [1, Th. 1]. We have for each of the theta constants, where we separate the $n_2 = 0$ terms from the rest,

$$1)\quad \theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \sum_{\substack{n_2=0\\ all\, n_1}} \exp\pi i\{\pi_{11}n_1^2\} + \sum_{\substack{n_2\neq 0\\ all\, n_1}} \exp\pi i\{\pi_{11}n_1^2 + 2\pi_{12}n_1n_2 + \pi_{22}n_2^2\},$$

$$2)\quad \theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} = \Sigma\exp\pi i\{\pi_{11}(n_1+\tfrac{1}{2})^2\} + \Sigma\exp\pi i\{\pi_{11}(n_1+\tfrac{1}{2})^2 + 2\pi_{12}(n_1+\tfrac{1}{2})n_2 + \pi_{22}n_2^2\},$$

$$3)\quad \theta\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \Sigma\exp\pi i\{\pi_{11}n_1^2\} + \Sigma\exp\pi i\{\pi_{11}n_1^2 + 2\pi_{12}n_1n_2 + \pi_{22}n_2^2 + n_2\},$$

$$4)\quad \theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \Sigma\exp\pi i\{\pi_{11}(n_1+\tfrac{1}{2})^2\} + \Sigma\exp\pi i\{\pi_{11}(n_1+\tfrac{1}{2})^2 + 2\pi_{12}(n_1+\tfrac{1}{2})n_2 + \pi_{22}n_2^2 + n_2\},$$

$$5)\quad \theta\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} = \Sigma\exp\pi i\left\{\begin{array}{l}\pi_{11}(n_1+\tfrac{1}{2})^2\\ +\pi_{12}(n_1+\tfrac{1}{2})\\ +\pi_{22}/4\end{array}\right\} + \Sigma\exp\pi i\,\{\pi_{11}(n_1+\tfrac{1}{2})^2 + 2\pi_{12}(n_1+\tfrac{1}{2})(n_2+\tfrac{1}{2}) + \pi_{22}(n_2+\tfrac{1}{2})^2\},$$

$$6)\quad \theta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} = \Sigma\exp\pi i\{\pi_{11}n_1^2 + \pi_{12}n_1 + \frac{\pi_{22}}{4}\} + \Sigma\exp\pi i\,\{\pi_{11}n_1^2 + 2\pi_{12}(n_1)(n_2+\tfrac{1}{2}) + \pi_{22}(n_2+\tfrac{1}{2})^2\}.$$

We note that the last two theta constants may be rewritten removing a factor, $\exp\pi i\pi_{22}/4$.

As $\pi_{22}\to i\infty$ we can readily compute that the first four expansions above tend to the appropriate theta constant in genus 1 with period π_{11}. The last two, aside from the factor $\exp\pi i\,\pi_{22}/4$, become, respectively,

$$\sum_{all\, n_1} \exp\pi i\{\pi_{11}(n_1+\tfrac{1}{2})^2 + \pi_{12}(n_1+\tfrac{1}{2})\}$$

and

$$\sum_{all\, n_1} \exp\pi i\{\pi_{11}n_1^2 + \pi_{12}n_1\}.$$

These two summations are distinct and not zero for general values of π_{11} and π_{12}.

An examination of the formulae for the branch points given in [1, Th. 1] shows that $1/\lambda_1$ and $1/\lambda_2$ coalesce to the same point distinct from $1/\lambda_3$, while $1/\lambda_3$ tends to

$$\frac{\theta^4\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0,\pi_{11})}{\theta^4\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\pi_{11})}.$$

We have, geometrically, the two branch points, $1/\lambda_1$ and $1/\lambda_2$, joined by a branch cut coalescing to each other. The limiting Riemann surface is then given by the branch points over 0, 1, ∞ and $1/\lambda_3$ with 0 and 1, and $1/\lambda_3$ and ∞, joined by branch cuts.

REMARK. To justify the assumption in the first paragraph of Section II, one computes the integrals dz/w and zdz/w, $w = \sqrt{z(1-z)(1-\lambda_1 z)(1-\lambda_2 z)(1-\lambda_3 z)}$, along the cycles of the homology basis and then one normalizes. When this is done the periods are given as functions of the branch points, and it is indeed the case that as $1/\lambda_1$ and $1/\lambda_2$ coalesce that $|\pi_{22}|$ goes to infinity, all other periods remaining finite.

OBSERVATION. If we apply a Siegel modular transformation to change the homology basis, we may compute the corresponding change in the period matrix We take for M, the matrix,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Then M acting on (γ, δ) gives (γ', δ') where $\gamma_1' = \gamma_1$, $\gamma_2' = \delta_2$, $\delta_1' = \delta_1$ and $\delta_2' = -\gamma_2$. The period matrix (π_{ij}) becomes, using $M \circ \pi = (A\pi + B)(C\pi + D)^{-1}$,

$$\begin{bmatrix} \pi_{11} - \dfrac{\pi_{12}^2}{\pi_{22}} & \dfrac{\pi_{12}}{\pi_{22}} \\ \dfrac{\pi_{12}}{\pi_{22}} & -\dfrac{1}{\pi_{22}} \end{bmatrix}.$$

We see that letting $\pi_{22} \to i\infty$ in our original degeneration corresponds to a matrix which has three entries tending to zero and the fourth tending to π_{11}.

III.

In a manner similar to what was done in Section II, we now degenerate S by letting π_{11} tend to $i\infty$. In this case we assume that for $|\pi_{11}|$ large the branch points over $1/\lambda_1$ $1/\lambda_2$, $1/\lambda_3$ are all near the branch point over ∞.

THEOREM 2. *Let S degenerate by letting π_{11} tend to $i\infty$. Then the branch points $1/\lambda_1$, $1/\lambda_2$, $1/\lambda_3$ all tend to ∞.*

REMARK. This degeneration may be viewed as the branch points over 0 and 1 tending to ∞, with the limit surface having branch points ove $0 = (1/\lambda_1)'$, $1 = (1/\lambda_2)'$, $(1/\lambda_3)'$ and ∞, as was done in [1, Section 5].

PROOF. Using an analysis analogous to the one in II, and the formulae for the branch points given in [1], we find that all the moveable branch points tend to ∞. Indeed, each formula for the branch points has theta constants in the denominator with characteristic whose first column is $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. For $\pi_{11} \to i\infty$ such theta constants go to 0.

Theorems 1 and 2 together give companion geometric results in the following sense. In Theorem 1 the degeneration corresponds to the shrinking of the γ_2 homology cycle, that is, the cycle surrounding the branch points over $1/\lambda_1$ and $1/\lambda_2$. By the remark after the statement of Theorem 2, this degeneration corresponds to the shrinking of the γ_1 homology cycle surrounding the originally fixed branch points over 0 and 1.

REFERENCES

1. A. Lebowitz, *Degeneration of a compact Riemann surface of genus* 2, Israel. J. Math. **12** (1972), 223–236.

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